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COMMENT

Comment on 'The Darboux transformation and algebraic deformations of shape-invariant potentials'

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Abstract
We show the equivalence of the recently formulated backward Darboux transformation of Gómez-Ullate *et al* and the Junker–Roy method of constructing isospectral Hamiltonians.

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The problem of enlarging the class of exactly solvable potentials in quantum mechanics has been studied by both physicists and mathematicians time and again. Families of Hamiltonians isospectral to a given exactly solvable Hamiltonian have been generated, either by inserting a new ground state, or deleting the original ground state, or maintaining an identical spectrum, by employing various techniques—the Darboux transformation [1] or the equivalent approach developed by Abraham and Moses [2], the factorization method of Infeld and Hull [3], the approach of supersymmetric (SUSY) quantum mechanics [4, 5] or that due to Pursey [6], etc.

In a recent paper, Gómez-Ullate, Kamran and Milson [8] investigate the backward Darboux transformation of shape-invariant potentials. By using the backward Darboux transformation they have obtained a number of non-shape-invariant exactly solvable potentials. On the other hand, a few years back, another formalism was developed by Junker and Roy [7], based on the SUSY formulation of one-dimensional systems, to construct a hierarchy of new families of the so-called conditionally exactly solvable (CES) systems, starting from known exactly solvable potentials [9]. Their approach is applicable to cases with both broken and unbroken SUSY.

In the present study our aim is to show the equivalence of the Junker–Roy [7] method and the backward Darboux transformation [8] method, taking the linear harmonic oscillator as an explicit example.

Let us start with the Hamiltonian (we follow the notation of [8])

$$\widehat{H} = -\partial_{xx} + \widehat{U}(x) \tag{1}$$

such that $\phi(x)$ is a formal eigenfunction of $\widehat{H}(x)$. Then the backward Darboux transformation $\widehat{U}(x)$ is given by [8]

$$U(x) = \widehat{U}(x) + 2\sigma_{xx} \tag{2}$$

where

$$\sigma = -\ln\phi. \tag{3}$$

The spectrum of $U(x)$ has an additional eigenvalue corresponding to the ground state eigenfunction

$$\psi_0 = \phi^{-1}. \tag{4}$$

The rest of the spectrum of $U(x)$ is identical to that of $\widehat{U}(x)$.

If the scenario is perceived in the framework of SUSY quantum mechanics, then the SUSY partner Hamiltonians $H_\pm$ defined by [7]

$$H_\pm(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{5}$$

are isospectral, except for a possible additional vanishing eigenvalue in one of the two Hamiltonians, $H_\pm$, in the case of unbroken SUSY.

The so-called SUSY partner potentials $V_\pm(x)$ are expressed in terms of the superpotential $W(x)$ as

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{6}$$

If $V_+(x)$ is an exactly solvable potential, then one can easily obtain the complete spectral properties of the partner $V_-(x)$ [7]. The point to be noted here is that $V_-(x)$ is not essentially shape invariant, but still exactly solvable.

One can take the following ansatz for the superpotential $W(x)$,

$$W(x) = W_0(x) + f(x) \tag{7}$$

where the superpotential $W_0(x)$ is chosen such that for $f = 0$ the corresponding partner potentials $V_\pm(x)$ belong to the known class of exactly solvable potentials.

If f is chosen such that it obeys the generalized Riccati equation

$$f^2(x) + 2W_0(x)f(x) + f'(x) = b \tag{8}$$

where b is an arbitrary real constant, then the partner potentials take the form

$$V_+(x) = W_0^2(x) + W_0'(x) + b \tag{9}$$

$$V_-(x) = W_0^2(x) - W_0'(x) + b - 2f'(x). \tag{10}$$

In the above expressions, b is an additive constant and $V_+(x)$ is taken to be exactly solvable. Choosing

$$f(x) = \frac{u'(x)}{u(x)} = \partial_x \ln u(x) \tag{11}$$

(8) reduces to

$$u''(x) + 2W_0(x)u'(x) - bu(x) = 0 \tag{12}$$

with the general solution as

$$u(x) = \alpha u_1(x) + \beta u_2(x). \tag{13}$$

In the case of unbroken SUSY, $V_-(x)$ has an additional ground state given by

$$\psi_0(x) = \exp\left(-\int W(x)\,\mathrm{d}x\right) = \frac{1}{u(x)}\exp\left(-\int W_0(x)\,\mathrm{d}x\right). \tag{14}$$

Putting $\psi_0^{-1} = \chi$, the equation satisfied by χ is found to be

$$-\chi'' + \left(W_0^2 + W_0' + b\right)\chi = 0 \tag{15}$$

i.e. χ is a formal eigenfunction of $V_+(x)$.

Thus if one identifies $V_+(x)$ with $\{\widehat{U}(x) + \beta\}$, where β is some constant, then

$$\chi \equiv \phi$$

so that

$$\begin{aligned} U(x) &= \widehat{U}(x) + 2\sigma_{xx} \\ &= \widehat{U}(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\phi \\ &= V_+(x) - \beta - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\phi \\ &= W_0^2 - W_0' + b - 2f' - \beta \\ &= V_-(x) - \beta. \end{aligned} \tag{16}$$

This proves the equivalence of the Junker–Roy approach [7] and the backward Darboux one [8].

Linear harmonic oscillator

In this section we demonstrate the equivalence of the two methods with the help of an explicit example, namely, the linear harmonic oscillator.

$$\widehat{U}(x) = x^2. \tag{17}$$

The formal eigenfunctions can be obtained from the solutions of equation (12) [7]. A set of simple formal eigenfunctions is of the form [7, 8]

$$\begin{aligned} \phi_k &= \frac{(-1)^k}{2^k\left(\frac{1}{2}\right)_k} H_{2k}(\mathrm{i}x)\,\mathrm{e}^{\frac{x^2}{2}} \\ &= \alpha_k u_k\,\mathrm{e}^{\frac{x^2}{2}} \end{aligned} \tag{18}$$

where $H_m(z)$ denote the generalized Hermite polynomials [10].

Hence from (2) and (3), the backward Darboux transform of $\widehat{U}(x)$ assumes the form

$$U^{(k)}(x) = x^2 - 2 - 32k^2\left\{\frac{H_{2k-1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}\right\}^2 + 16k(2k-1)\frac{H_{2k-2}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}. \tag{19}$$

Since

$$u_k = H_{2k}(\mathrm{i}x) \tag{20}$$

f turns out to be

$$\begin{aligned} f &= \frac{4\mathrm{i}kH_{2k-1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} \\ &= \sum_{i=1}^{k}\frac{2gx}{1+gx^2} \end{aligned} \tag{21}$$

and

$$b = 4k \qquad \beta = b + 1 \tag{22}$$

giving the following superpotentials for successive orders:

$k = 1$

$$W(x) = x + \frac{2gx}{1 + gx^2} \qquad g = 2 \tag{23}$$

$k = 2$

$$W(x) = x + \frac{2g_1 x}{1 + g_1 x^2} + \frac{2g_2 x}{1 + g_2 x^2} \tag{24}$$

with

$$g_1 = 2 + \tfrac{2}{3}\sqrt{6} \tag{25}$$

$$g_2 = 2 - \tfrac{2}{3}\sqrt{6} \tag{26}$$

proving that if one starts with the backward Darboux transformation [8], one can reproduce all the results of the CES potentials of Junker and Roy [7] and vice versa, showing the equivalence of the two methods. Analogous analyses hold for the Morse, the hyperbolic Pöschl–Teller and other potentials.

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References

[1] Darboux G 1982 *C. R. Acad. Sci., Paris* **94** 1456
Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)
[2] Abraham P B and Moses H E 1980 *Phys. Rev.* A **22** 1333
[3] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 28
[4] Witten E 1981 *Nucl. Phys.* B **188** 513
[5] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
[6] Pursey D L 1986 *Phys. Rev.* D **33** 1048
Pursey D L 1986 *Phys. Rev.* D **33** 2267
Pursey D L 1987 *Phys. Rev.* D **36** 1103
[7] Junker G and Roy P 1998 *Ann. Phys.* **270** 155
Junker G and Roy P 1997 *Phys. Lett.* A **232** 155
[8] Gómez-Ullate D, Kamran N and Milson R 2004 *J. Phys. A: Math. Gen.* **37** 1789
[9] Gendenshtein L E 1983 *JETP Lett.* **38** 356
[10] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)